

An A-Stable Uniform Order Six Linear Multi-Step Method for Testing Stiffly First Order Ordinary Differential Equations.

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ABSTRACT: *This paper formulates an A-Stable uniform order six linear multi-step method for testing stiffly first order ordinary differential equations. Power series approximate solution was used as the basic function using interpolation and collocation techniques to derive the continuous schemes which was found to converge. The developed method is then applied to solve the system of first-order stiffly ordinary differential equations and the result is displaced side by side in tables with the existing one, the method tends to give better approximation when implemented numerically than the existing method with which we compared our results.*

KEYWORDS: *A – stable, Linear Multi-step method, Stiffly ODEs, Interpolation and Collocation.*

I. INTRODUCTION

In the early 1950s, as a result of some pioneering work by (Curtiss, C. F. and Hirschfelder, J. O., 1952), (Stuart, A. M. and Humphries, A. R., 1996) was realized that there was an important class of ordinary differential equations (ODEs), which have become known as stiff equations, which presented a severe challenge to numerical methods that existed at that time. Since then an enormous amount of effort has gone into the analysis of stiff problems and, as a result, a great many numerical methods have been proposed for their solution. More recently, however, there have been some strong indications that the theory which underpins stiff computation is now quite well understood, and, in particular, the excellent text of (Hairer, E. and Wanner, G., 1996) has helped put this theory on a firm basis. As a result of this, some powerful codes have now been developed and these can solve quite difficult problems in a routine and reliable way.

Interestingly, differential equations arising from the modeling of physical phenomena often do not have exact solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary. To that extent, several numerical methods such as finite difference methods, finite element methods and finite volume methods, among others, have been developed based on the nature and type of the differential equation to be solved.

A differential equation can be classified into Ordinary Differential Equation (ODE), Partial Differential Equation (PDE), Stochastic Differential Equation (SDE), Impulsive Differential Equation (IDE), Delay Differential Equation (DDE), etc. (Stuart, A. M. and Humphries, A. R., 1996). The main purpose of this paper is to outline some of the important theory behind stiff computation and to direct users of numerical software to those codes which are most likely to be effective for their particular problem. In sciences and engineering, mathematical models are formulated to aid in the understanding of physical phenomena. The formulated model often yields an equation that contains the derivatives of an unknown function. Such an equation is referred to as Differential equation, (Fatunla, S. O., 1988). As opposed to one-step methods, which only utilize one previous value of the numerical solution to approximate the subsequent value, multistep methods approximate numerical values of the solution by referring to more than one previous value (Skwame, Y., Sabo, J. and Kyagya, T. Y., 2017). Accordingly, multistep methods may often achieve greater accuracy than one-step methods that use the same number of function evaluations, since they utilize more information about the known portion of the solution than one-step methods do. The methods of Euler, Heun, Taylor and Runge-Kutta are called single-step methods because they use only the information from one previous point to compute the successive point, that is, only the initial point (t_0, y_0) is used to compute (t_1, y_1) and in general Y_k is needed to compute Y_{k+1} . In this article, the system of first-order ODEs of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \tag{1}$$

Is considered. This research is organized as follows: in the coming section, we carried out the derivation of the method, where we considered one-step through interpolation and collocation at first derivatives.

The details of the analysis of the method which include order, error constant, consistency, zero stability and stability region were discussed in Section three. In the fourth section, some numerical problems were solved and the performance of the developed method was compared with those of the existing methods, (Skwame, Y., Kumleng, G. M. and Bakari, I. A., 2017, Skwame, Y., 2018, Skwame, Y., Sunday, J. and Sabo, J. 2018). Finally, the conclusion was drawn in section five.

II. THEORETICAL PROCEDURE

In this section, the formation of single-step continuous uniform order six linear multi-step method for testing stiffly first order ordinary differential equations (1) is developed following (Alkasassbeh, M. and Zurni, O., 2017, Omar, Z. and Sulaiman, M., 2004).

The general solution for solving (1) is derived in this section. Let the power series of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x-x_n}{h} \right)^i, \quad (2)$$

be the approximate solution to (1) for $x \in [x_n, x_{n+1}]$ where $n = 0, 1, 2, \dots, N-1$, a_i 's are the real coefficients to be determined, v is the number of collocation points, m is the number of interpolation points and $h = x_n - x_{n-1}$ is a constant step size of the partition of interval $[a, b]$, which is given by

$$a = x_0 < x_1 < \dots < x_N = b$$

Differentiating (2) once yields:

$$y'(x) = \sum_{i=1}^{v+m-1} \frac{ia_i}{h} \left(\frac{x-x_n}{h} \right)^{i-1} \quad (3)$$

Interpolating (2) at the point x_n and collocating (3) at all points in the selected interval, i.e., $x_n, x_{n+\frac{1}{5}}, x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}}, x_{n+\frac{4}{5}}$ and x_{n+1} , gives the following nonlinear equations which can be written in matrix form as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & \frac{2}{5} & \frac{3}{25} & \frac{4}{125} & \frac{1}{125} & \frac{6}{3125} \\ 0 & \frac{1}{h} & \frac{4}{5} & \frac{12}{25} & \frac{32}{125} & \frac{16}{125} & \frac{192}{3125} \\ 0 & \frac{1}{h} & \frac{6}{5} & \frac{27}{25} & \frac{108}{125} & \frac{81}{125} & \frac{1458}{3125} \\ 0 & \frac{1}{h} & \frac{8}{5} & \frac{48}{25} & \frac{256}{125} & \frac{256}{125} & \frac{6144}{3125} \\ 0 & \frac{1}{h} & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix} \quad (4)$$

Applying the Gaussian elimination method on (4) gives the coefficient a_i 's, for $i = 0(1)7$.

These values are then substituted into (2) to give the implicit continuous hybrid method of the form:

$$y(x) = \sum_{i=0} \alpha_i(x) y_{n+i} + h \sum_{i=0}^1 \beta_i(x) f_{n+i} + h \sum_{i=k} \beta_i(x) f_{n+k} \quad k = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1 \quad (5)$$

On evaluating (5) yield the following continuous schemes,

$$\left. \begin{aligned} \alpha_0 &= 0 \\ \beta_{\frac{1}{5}} &= \frac{25}{288} t^2 h (144 - 616t + 1065t^2 - 840t^3 + 250t^4) \\ \beta_{\frac{2}{5}} &= \frac{25}{144} t^2 h (72 - 428t + 885t^2 - 780t^3 + 250t^4) \\ \beta_{\frac{3}{5}} &= -\frac{25}{144} t^2 h (-48 - 312t + 735t^2 + 470t^3) \\ \beta_{\frac{4}{5}} &= -\frac{25}{288} t^2 h (36 - 244t + 615t^2 - 660t^3 + 250t^4) \\ \beta_1 &= \frac{1}{288} t^2 h (50t^2 + 40t + 9)(-4 + 5t)^2 \end{aligned} \right\} \quad (6)$$

evaluate (6) at both grid and off-grid points yield the five continuous schemes as follows

$$\begin{aligned} y_{n+\frac{1}{5}} &= y_n + \frac{1}{7200} h \left(475f_n + 1427f_{n+\frac{1}{5}} - 798f_{n+\frac{2}{5}} + 482f_{n+\frac{3}{5}} - 173f_{n+\frac{4}{5}} + 27f_{n+1} \right) \\ y_{n+\frac{2}{5}} &= y_n + \frac{1}{450} h \left(28f_n + 129f_{n+\frac{1}{5}} + 14f_{n+\frac{2}{5}} + 14f_{n+\frac{3}{5}} - 6f_{n+\frac{4}{5}} + f_{n+1} \right) \\ y_{n+\frac{3}{5}} &= y_n + \frac{3}{800} h \left(17f_n + 73f_{n+\frac{1}{5}} + 38f_{n+\frac{2}{5}} + 38f_{n+\frac{3}{5}} - 7f_{n+\frac{4}{5}} + f_{n+1} \right) \\ y_{n+\frac{4}{5}} &= y_n + \frac{2}{225} h \left(7f_n + 32f_{n+\frac{1}{5}} + 12f_{n+\frac{2}{5}} + 32f_{n+\frac{3}{5}} + 7f_{n+\frac{4}{5}} \right) \\ y_{n+1} &= y_n + \frac{1}{288} h \left(19f_n + 75f_{n+\frac{1}{5}} + 50f_{n+\frac{2}{5}} + 50f_{n+\frac{3}{5}} + 75f_{n+\frac{4}{5}} + 19f_{n+1} \right) \end{aligned} \quad (7)$$

III. CONVERGENCE ANALYSIS

In this section, the basic properties of (7) shall be analyzed.

Order and error Constants of the Methods

Given linear difference operator (Butcher, J. C., 2009)

$$\ell[y(x), h] = \sum_{j=0}^k [(a_j y(x + jh) - h\beta_j y'(x + jh))]$$

and using Taylor expansion about the point x, we get

$$\ell[y(x), h] = c_0 y(x) + c_1 h y'(x) + \cdots + c_q h^q y^{(q)}(x) + \cdots \quad (6)$$

where

$$c_0 = \alpha_0 + \alpha_1 + \dots + \alpha_k$$

$$c_1 = \alpha_1 + 2\alpha_2 + \dots + k\alpha_k - (\beta_0 + \beta_1 \dots \beta_k)$$

\vdots

$$c_q = \frac{1}{q!}(\alpha_1 + 2^q \alpha_2 + \dots + k^q \alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{(q-1)} \beta_2 + \dots + k^{(q-1)} \beta_k), \quad q = 2, 3, \dots$$

Such that when $c_0 = c_1 = \dots = c_p = 0$ and $c_{p+1} \neq 0$ of (6) then

C_{p+1} is the Error constant, and p is the order of LMM.

According to (Butcher, J. C., 2009), the order of the new method is obtained by using the Taylor series and it is found that the developed method (7) is of uniformly order Six, with an error constants vector of:

$$C_6 = [-1.8265 \times 10^{-7}, -1.2529 \times 10^{-7}, -1.6574 \times 10^{-7}, -1.0836 \times 10^{-7}, -2.9101 \times 10^{-7}]^T$$

Consistency

Definition 3.1: The hybrid block method (7) is said to be consistent if it has an order more than or equal to one i.e. $P \geq 1$. Therefore, the new method is consistent (Dahlquist, G., 1956).

Zero Stability

Definition 3.2: The hybrid block method (7) is said to be zero stable if the first characteristic polynomial $\pi(r)$

having roots such that $|r_z| \leq 1$ and if $|r_z| = 1$, then the multiplicity of r_z must not greater than two (Dahlquist, G., 1956 and Butcher, J. C., 2009).

In order to find the zero-stability of hybrid block method (7), we only consider the first characteristic polynomial of the method according to definition (3.2) as follows

$$\Pi(r) = r \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = r^4(r-1)$$

which implies $r = 0, 0, 0, 0, 1$. Hence the method is zero-stable since $|r_z| \leq 1$ and if $|r_z| = 1$.

Convergence

Theorem (3.1): Consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the method is consistent and zero stable, it implies the method is convergent for all point (Butcher, J. C., 2009).

Regions of Absolute Stability (RAS)

Definition 3.3: The Region of Absolute Stability is the set of all points $z \in Z$ such that all roots of characteristic equation are of absolute value less than one, (Lambert, (1973). According to (Dahlquist, G., 1956 and Lambert, J. D., 1991) the absolute stability region of the new method is shown in the figure below.

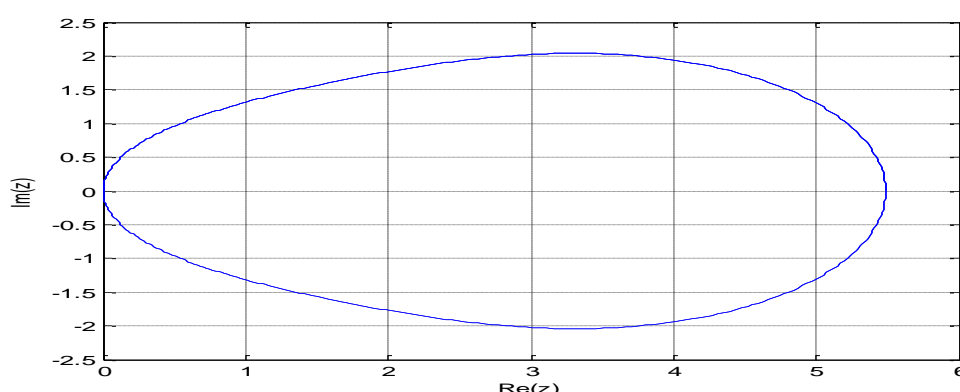


Figure 3.1: Region of Absolute Stability

IV. THE IMPLEMENTATION OF METHOD

In this section, the efficiency and the performance of the new method (7) is investigated with four systems first order stiffly ODEs. The problems considered are the ones widely used by (Skwame, Y., Sabo, J. and Kyagya, T. Y., 2017, Sabo, J., Skwame, Y., Kyagya, T. Y., (2018), Akinfenwa, O. A., Abdulganiy, R. I., Akinnukawe, B. I., Okunuga, S. A. and Rufai, U. O., 2017), and the results obtained shall be compared with the existing method in terms of error.

Problem Consider the stiffly problem,

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad 0 \leq x \leq 0.8, \quad h = 0.1$$

$$y(x) = \frac{1}{2} \begin{pmatrix} e^{-2x} + e^{-40x} (\cos(40x) + \sin(40x)) \\ e^{-2x} - e^{-40x} (\cos(40x) + \sin(40x)) \\ 2e^{-40x} (\sin(40x) - \cos(40x)) \end{pmatrix}$$

(Source: Skwame, Y., Sabo, J. and Kyagya, T. Y., 2017)

Table 4.1: Comparison of result of new method with that of (Skwame, Y., Sabo, J. and Kyagya, T. Y., 2017)

| $x - \text{values}$ | Error in Skwame, <i>et-al.</i> , (2017) | | | Error in new method | | |
|---------------------|-----------------------------------------|-----------------------|------------------------|------------------------|------------------------|------------------------|
| | y_1 | y_2 | y_3 | y_1 | y_2 | y_3 |
| 0.1 | 2.23×10^{-2} | 2.23×10^{-2} | 2.53×10^{-2} | 2.15×10^{-2} | 2.15×10^{-2} | 2.42×10^{-2} |
| 0.2 | 1.06×10^{-4} | 9.14×10^{-5} | 1.68×10^{-4} | 9.65×10^{-5} | 9.65×10^{-5} | 9.85×10^{-5} |
| 0.3 | 8.23×10^{-6} | 9.10×10^{-6} | 1.33×10^{-5} | 4.22×10^{-7} | 4.21×10^{-7} | 1.24×10^{-7} |
| 0.4 | 9.60×10^{-6} | 9.30×10^{-6} | 1.60×10^{-7} | 1.30×10^{-7} | 1.30×10^{-7} | 1.35×10^{-7} |
| 0.5 | 9.67×10^{-6} | 9.67×10^{-6} | 1.68×10^{-9} | 6.00×10^{-10} | 2.00×10^{-10} | 5.89×10^{-10} |
| 0.6 | 9.50×10^{-6} | 9.50×10^{-6} | 9.12×10^{-11} | 4.00×10^{-10} | 0.00×10^0 | 3.75×10^{-11} |
| 0.7 | 9.08×10^{-6} | 9.08×10^{-6} | 1.05×10^{-10} | 4.00×10^{-10} | 0.00×10^0 | 2.59×10^{-10} |
| 0.8 | 8.49×10^{-6} | 8.49×10^{-6} | 6.84×10^{-11} | 4.00×10^{-10} | 0.00×10^0 | 1.30×10^{-10} |

Problem 4.2

Consider the stiffly problem, $y_1' = -100y_1 + 9.901y_2$; $y_1(0) = 1$
 $y_2' = 0.1y_1 - y_2$; $y_2(0) = 10$, $h = 0.1$

With Exact Solution

$$y_1(x) = e^{-0.99x}$$

$$y_2(x) = 10e^{-0.99x}$$

$$x \in [0, 1]$$

(Source, Sabo, J., *et-al*, 2018)

Table 4.2: Comparison of absolute error with that of Sabo, *et-al.*, (2018)

| $x - \text{values}$ | Error in Sabo, <i>et-al.</i> , (2018) | | Error in new method | |
|---------------------|---------------------------------------|-----------------------|------------------------|-----------------------|
| | $y_1(x)$ | $y_2(x)$ | $y_1(x)$ | $y_2(x)$ |
| 0.1 | 1.80×10^{-9} | 1.40×10^{-8} | 5.00×10^{-10} | 4.00×10^{-9} |
| 0.2 | 2.70×10^{-9} | 2.30×10^{-8} | 1.50×10^{-9} | 7.00×10^{-9} |
| 0.3 | 3.70×10^{-9} | 3.30×10^{-8} | 1.50×10^{-9} | 9.00×10^{-9} |
| 0.4 | 4.40×10^{-9} | 3.90×10^{-8} | 2.20×10^{-9} | 1.30×10^{-8} |
| 0.5 | 5.00×10^{-9} | 4.70×10^{-8} | 2.20×10^{-9} | 1.50×10^{-8} |
| 0.6 | 5.20×10^{-9} | 5.00×10^{-8} | 2.30×10^{-9} | 1.30×10^{-8} |
| 0.7 | 5.40×10^{-9} | 5.20×10^{-8} | 1.30×10^{-9} | 1.20×10^{-8} |
| 0.8 | 5.70×10^{-9} | 5.40×10^{-8} | 2.10×10^{-8} | 1.40×10^{-8} |
| 0.9 | 5.60×10^{-9} | 5.50×10^{-8} | 1.70×10^{-9} | 1.50×10^{-8} |
| 1.0 | 5.70×10^{-9} | 5.50×10^{-8} | 2.40×10^{-9} | 1.70×10^{-8} |

Problem 4.3

Consider the stiffly problem,

$$y_1' = -y_1; \quad y_1(0) = 1$$

$$y_2' = -2000y_2; \quad y_2(0) = 1 \quad h = 0.1, \quad 0 \leq x \leq 1$$

$$y_1(x) = e^{-x}$$

$$y_2(x) = e^{-2000x}$$

Table 4.3: Comparison of absolute error

| $x - \text{values}$ | y_1 | y_2 |
|---------------------|------------------------|-----------------------|
| 0.1 | 2.00×10^{-10} | 7.96×10^{-1} |
| 0.2 | 1.00×10^{-10} | 6.34×10^{-1} |
| 0.3 | 3.00×10^{-10} | 5.04×10^{-1} |
| 0.4 | 5.00×10^{-10} | 4.01×10^{-1} |
| 0.5 | 5.00×10^{-10} | 3.19×10^{-1} |
| 0.6 | 5.00×10^{-10} | 2.54×10^{-1} |
| 0.7 | 5.00×10^{-10} | 2.02×10^{-1} |
| 0.8 | 5.00×10^{-10} | 1.61×10^{-1} |
| 0.9 | 5.00×10^{-10} | 1.28×10^{-1} |
| 1.0 | 4.00×10^{-10} | 1.02×10^{-1} |

Problem 4.4 Consider the stiffly problem,

$$y_1' + 0.1y_1 + 49.9y_2 = 0; \quad y_1(0) = 2$$

$$y_2' + 50y_2 = 0; \quad y_2(0) = 0, \quad h = \frac{1}{1000}$$

$$y_3' - 70y_2 + 120y_3 = 0; \quad y_3(0) = 2$$

With Exact Solution

$$y_1(x) = e^{-50x} + e^{-0.1x}$$

$$y_2(x) = e^{-50x}$$

$$y_3(x) = e^{-50x} + e^{-120x}$$

$$x \in [0, 1]$$

(Source, Akinfenwa, O. A., *et-al*, 2017)

Table 4.4: Comparison of absolute error

| $x - values$ | Absolute error | | |
|--------------|-----------------------|------------------------|-----------------------|
| | y_1 | y_2 | y_3 |
| 0.1 | 9.85×10^{-3} | 5.00×10^{-10} | 8.87×10^{-1} |
| 0.2 | 1.96×10^{-2} | 1.00×10^{-9} | 7.40×10^{-1} |
| 0.3 | 2.93×10^{-2} | 1.40×10^{-9} | 6.98×10^{-1} |
| 0.4 | 3.88×10^{-2} | 1.70×10^{-9} | 6.19×10^{-1} |
| 0.5 | 4.83×10^{-2} | 2.00×10^{-9} | 5.49×10^{-1} |
| 0.6 | 5.76×10^{-2} | 2.30×10^{-9} | 4.87×10^{-1} |
| 0.7 | 6.70×10^{-2} | 2.50×10^{-9} | 4.32×10^{-1} |
| 0.8 | 6.61×10^{-2} | 2.70×10^{-9} | 3.83×10^{-1} |
| 0.9 | 8.52×10^{-2} | 2.90×10^{-9} | 3.40×10^{-1} |
| 1.0 | 9.42×10^{-2} | 3.10×10^{-9} | 3.01×10^{-1} |

V. CONCLUSIONS

The single-step continuous uniform order six linear multi-step method for testing stiffly first order ordinary differential equations has been developed in this paper. The analysis of the new method was studied and it was found to be consistent, convergent and zero-stable, with the region of absolute stable within which the method is stable. The newly constructed method was applied to solve four systems of first-order stiffly ordinary differential equations and from the results obtained, it was clear that the developed method performed better. Therefore, the general solution of first order Linear Multistep Method (LMM) is a convenient technique for determining the solutions of mathematical modeling since it can approximate the result even though the efficiency is less than the other multistep method.

Recommendation: The pair of numerical methods developed in this paper is recommended for testing first order stiffly ordinary differential equations. The basis function (power series) used is also recommended for the derivation of Numerical methods for first order differential equations and the pair of methods derived are also recommended for the solution of systems of first order stiffly ordinary differential equations.

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